Truss Analysis Discussion and Interpretation Using Linear Systems of Equalities and Inequalities

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Abstract

This paper shows the complementary roles of mathematical and engineering points of view when dealing with truss analysis problems involving systems of linear equations and inequalities. After the compatibility condition and the mathematical structure of the general solution of a system of linear equations is discussed, the truss analysis problem is used to illustrate its mathematical and engineering multiple aspects, including an analysis of the compatibility conditions and a physical interpretation of the general solution, and the generators of the resulting affine space. Next, the compatibility and the mathematical structure of the general solution of linear systems of inequalities are analyzed and the truss analysis problem revisited adding some inequality constraints, and discussing how they affect the resulting general solution and many other aspects of it. Finally, some conclusions are drawn.

Key words: Compatibility, cones, dual cones, linear spaces, polytopes, simultaneous solutions, truss analysis.
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1 Introduction

There are many engineering problems that involve linear systems of equalities and inequalities. Engineers are familiar with systems of equalities with a unique solution, but many of them are not used to deal with systems of equations with multiple solutions and their interpretations. The problem is even worse when systems of inequalities are dealt with, because only few people know how to obtain the associated compatibility conditions and solve them. These two types of systems can be interpreted from the mathematical or the engineering points of view, which are complementary and provide a deep understanding of the problem under study. However, people working in these problems use to have knowledge about only one of these two perspectives and are unaware of the relations between the mathematical and the engineering concepts, which leads to important limitations in the capacity of extracting conclusions from the results that can be expected after a careful analysis of these problems from both points of view.

This paper points out these relations and makes them explicit for the readers to discover the new world that arises when contemplating the compatibility conditions or the set of general solutions from this dual perspective.

In this paper, we have selected a particular example to illustrate these two points of view, the truss analysis problem, and we exploit this dual (mathematical and engineering) perspective to deal with a problem that involves linear systems of equalities or inequalities, depending on the constraints used to model the reality. As we shall see, many questions of practical interest arise and can be answered thanks to this dual analysis of the problem.

The paper is structured as follows. In Section 2 the problem of determining the compatibility conditions of systems of linear equalities and solving them together with an analysis of the general mathematical structure of their solutions is discussed. In Section 3 the truss analysis problem is described and used to illustrate all the theoretical methods. In Section 4 we classify truss structures according to some mathematical criteria in isostatic, hyperstatic, mechanism and critical trusses. In Section 5 we discuss the compatibility of systems of linear inequalities and the mathematical structure of their solutions. In Section 6 we revisit the truss analysis problem adding some constraints to illustrate the methodology and several engineering problems. Finally, in Section 7 some conclusions and recommendations are given.

2 Dealing with Systems of Equations

In many engineering applications we find systems of linear equations of the form:

$$\mathbf{A}\mathbf{x} = \mathbf{b}, \tag{1}$$

where $\mathbf{A} \in \mathbb{R}^{m \times n}$, and $\mathbf{b} \in \mathbb{R}^m$, being m and n the number of equations and unknowns, respectively, which can be equal or not. Before trying to solve a system of equations of this type it is interesting to check whether or not the system is compatible, i. e., if it has some solutions. These conditions can be given in terms of $b_1, b_2, \ldots, b_m$, and always have a physical or engineering meaning that raises some light about the problem under consideration.

Once the system is proven to have solution we can obtain the set of all its possible solutions. To solve these two problems an algorithm that gives the linear space orthogonal to the linear space generated by a set of vectors can be used (for a detailed description of this algorithm see Castillo, Cobo, Jubete, Pruneda and Castillo [3], Castillo, Cobo, Fernández-Canteli, Jubete and Pruneda [1], Cobo, Jubete and Pruneda [2]).

2.1 Deciding whether or not a linear system of equations is compatible

The system (1) can be written as

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n = \mathbf{b}, \tag{2}$$

which shows that the vector $\mathbf{b} = (b_1, \ldots, b_m)^T$ belongs the linear space generated by the column vectors $\{\mathbf{a}_1, \mathbf{a}_1, \ldots, \mathbf{a}_n\}$ of the system matrix $\mathbf{A}$, i.e., the compatibility requires:

$$\mathbf{b} \in \mathcal{L}(\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n) \Leftrightarrow \mathbf{b} \in \left(\mathcal{L}(\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n)^\perp\right)^\perp,$$

where $\perp$ refers to the orthogonal set. Thus, analyzing the compatibility of the system of equations (1) reduces to finding the linear subspace $\mathcal{L}\{\mathbf{w}_1, \ldots, \mathbf{w}_p\}$ orthogonal to $\mathcal{L}\{\mathbf{a}_1, \ldots, \mathbf{a}_n\}$ and checking whether or not $\mathbf{b}^T\mathbf{W} = \mathbf{0}$.

Example 1 (Compatibility of a linear system of equations) Suppose that we are interested in determining the conditions under which the system of

equations

$$
\begin{aligned}
x_1 \ +x_2 \ -x_3 \ +x_4 \ +x_5 &= a \\
-x_2 \ +2x_3 \ +x_4 \ -2x_5 &= b \\
x_1 \ -x_2 \ +2x_3 \qquad\qquad &= c \\
-2x_1 \ +2x_2 \ -3x_3 \ +x_4 \ -x_5 &= d \\
x_1 \ +x_2 \ +x_3 \ +x_4 \ +x_5 &= e
\end{aligned}
\tag{3}
$$

is compatible. Then, first we obtain the linear subspace orthogonal to the linear subspace generated by the column vectors in (3) that is:

$$
\mathbf{W} = \mathcal{L}\{\mathbf{w}_1\} = \mathcal{L}\left\{(0, 2, -7, -3, 1)^T\right\},
\tag{4}
$$

which implies the following compatibility condition:

$$
\mathbf{w}_1^T(a, b, c, d, e)^T = (0, 2, -7, -3, 1)(a, b, c, d, e)^T = 0 \Rightarrow 2b - 7c - 3d + e = 0.
\tag{5}
$$

2.2 Solving a homogeneous system of linear equations

Consider the homogeneous system of equations

$$
\mathbf{Ax} = \mathbf{0},
\tag{6}
$$

which can be written as

$$
\mathbf{a}^i\mathbf{x}^T = 0; \ i = 1, \ldots, m.
\tag{7}
$$

Expression (7) shows that $(x_1, \ldots, x_n)$ is orthogonal to the set of row vectors $\{\mathbf{a}^1, \mathbf{a}^2, \ldots, \mathbf{a}^m\}$ of $\mathbf{A}$.

Then, obtaining the solution to system (7) reduces to determining the linear subspace orthogonal to the linear subspace generated by the rows of matrix $\mathbf{A}$. Thus, the general solution of an homogeneous system of linear equations is a linear space, i.e., of the form

$$
\mathbf{x} = \sum_{i=1}^{p} \rho_i \mathbf{v}_i; \quad \rho_i \in \mathbb{R}.
$$

Example 2 (An homogeneous system of linear equations) The system

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of linear equations

$$\begin{aligned}
x_1 \qquad\quad +x_3 \ -2x_4 \ +x_5 &= 0 \\
x_1 \ -x_2 \ -x_3 \ +2x_4 \ +x_5 &= 0 \\
-x_1 \ +2x_2 \ +2x_3 \ -3x_4 \ +x_5 &= 0
\end{aligned} \tag{8}$$

has as general solution the linear space

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \rho_1 \begin{pmatrix} 1 \\ 2 \\ -3 \\ -1 \\ 0 \end{pmatrix} + \rho_2 \begin{pmatrix} 0 \\ 2 \\ -7 \\ -3 \\ 1 \end{pmatrix} ; \quad \rho_1, \rho_2 \in \mathbb{R}.$$

2.3 Solving a complete system of linear equations

Now consider again the complete system of linear equations (1), that adding the artificial variable x_{n+1}, can be written as

$$\begin{aligned}
a_{11}x_1 \ +a_{12}x_2 \ +\cdots \ +a_{1n}x_n \ -b_1 x_{n+1} &= 0, \\
a_{21}x_1 \ +a_{22}x_2 \ +\cdots \ +a_{2n}x_n \ -b_2 x_{n+1} &= 0, \\
\cdots \quad \cdots \quad \cdots \quad \cdots \qquad\quad \cdots \quad \cdots & \\
a_{m1}x_1 \ +a_{m2}x_2 \ +\cdots \ +a_{mn}x_n \ -b_m x_{n+1} &= 0, \\
x_{n+1} &= 1.
\end{aligned} \tag{9}$$

The first m equations of the system (9) can be written as

$$\begin{aligned}
(a_{11}, \ldots, a_{1n}, -b_1)(x_1, \ldots, x_n, x_{n+1})^T &= 0, \\
(a_{21}, \ldots, a_{2n}, -b_2)(x_1, \ldots, x_n, x_{n+1})^T &= 0, \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \quad \cdots & \\
(a_{m1}, \ldots, a_{mn}, -b_m)(x_1, \ldots, x_n, x_{n+1})^T &= 0,
\end{aligned} \tag{10}$$

which shows that $(x_1, \ldots, x_n, x_{n+1})$ is orthogonal to the set of vectors

$$\{(a_{11}, \ldots, a_{1n}, -b_1), (a_{21}, \ldots, a_{2n}, -b_2), \ldots, (a_{m1}, \ldots, a_{mn}, -b_m)\},$$

i.e., the solution of (10) is the linear subspace orthogonal to the linear subspace generated by the rows of matrix $\mathbf{A}_b$, i.e.,

$$\mathcal{L}\{(a_{11}, \ldots, a_{1n}, -b_1), (a_{21}, \ldots, a_{2n}, -b_2), \ldots, (a_{m1}, \ldots, a_{mn}, -b_m)\}^{\perp}.$$

Thus, the solution of (1) is the projection on $X_1 \times \cdots \times X_n$ of the intersection of such a orthogonal linear subspace and the set $\{\mathbf{x}|x_{n+1} = 1\}$. In other words, the general solution of a complete system of linear equations is an affine space, that is the sum of a constant vector plus a linear space:

$$\mathbf{x} = \mathbf{x}^0 + \sum_{i=1}^{p} \rho_i \mathbf{v}_i; \quad \rho_i \in \mathbb{R},$$

where the first vector in the right hand side is an arbitrary particular solution of the system (1), and the linear space is the set of solutions of the associated homogeneous system.

Example 3 (A complete system of linear equations) The system of linear equations

$$
\begin{aligned}
x_1 && +x_3 &-2x_4 &+x_5 &= 1 \\
x_1 &-x_2 &-x_3 &+2x_4 &+x_5 &= 2 \\
-x_1 &+2x_2 &+2x_3 &-3x_4 &+x_5 &= 1
\end{aligned}
\tag{11}
$$

has as general solution the affine space

$$
\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} + \rho_1 \begin{pmatrix} 1 \\ 2 \\ -3 \\ -1 \\ 0 \end{pmatrix} + \rho_2 \begin{pmatrix} 0 \\ 2 \\ -7 \\ -3 \\ 1 \end{pmatrix}; \quad \rho_1, \rho_2 \in \mathbb{R}.
$$

3   The Truss Analysis Problem

Consider the engineering problem of structures consisting of trusses, i.e., structures made of bars joined by frictionless hinges which imply that only tension or compression stresses exist in the bars. A typical truss structure is shown in Figure 1.

With reference to the bar element (see Figure 2) that has length L and cross-section A, let us first establish the stiffness k. By definition, the force N is
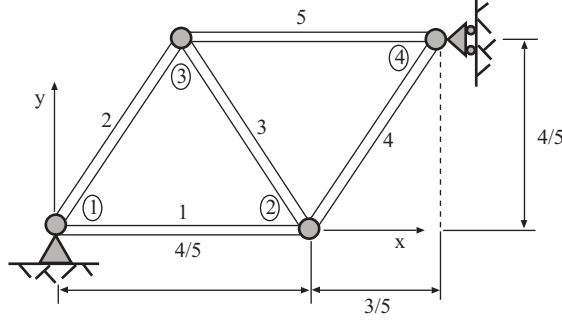
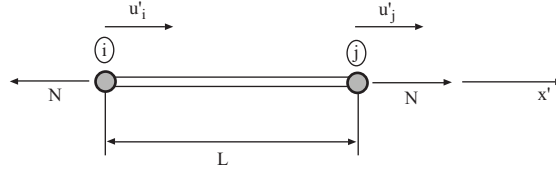Fig. 1. Two-dimensional truss structure consisting of bars joined by frictionless hinges.

Fig. 2. Loads acting on an individual bar.

related to the normal stress σ as

$$N = A\sigma, \tag{12}$$

and assuming that the bar behavior is linearly elastic, according to Hooke's law, we have

$$\sigma = E\epsilon, \tag{13}$$

where E is the Young's modulus and ϵ is the longitudinal strain, that is assumed constant along its length, and has value

$$\epsilon = \frac{u'_j - u'_i}{L}, \tag{14}$$

where u'_i and u'_j are the longitudinal displacements of the end points i and j, respectively.

Combining Equations (12)-(14) we obtain

$$N = k(u'_j - u'_i), \tag{15}$$

where $k = AE/L$ is the bar stiffness.

To calculate a given truss structure first we need to know the behavior of a single bar in an arbitrary direction. Thus, consider the bar shown in Figure 3 (a). Our purpose is to determine the relation between the node forces $F_{x_i}, F_{y_i}, F_{x_j}, F_{y_j}$ and the node displacements u_i, v_i, u_j, v_j.

To this aim, we introduce a local x'-y'-coordinate system (see Figure 3 (b)) with its origin at the end point i and its axis x' directed along the bar axis
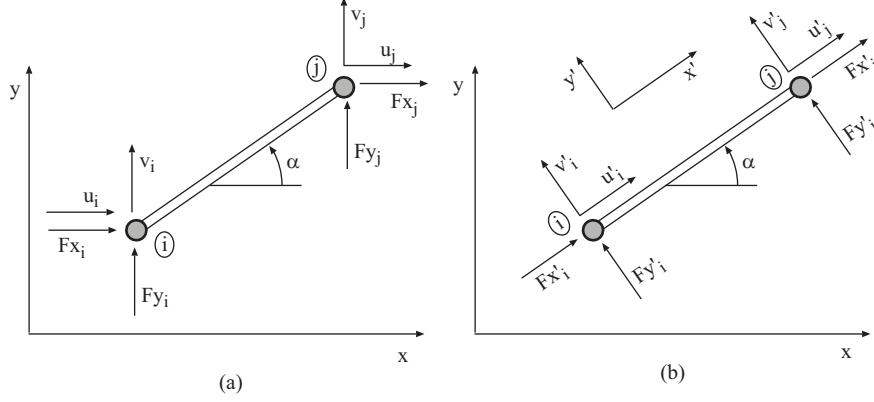
Fig. 3. Displacements and force components of a bar element with respect to: (a) the global x-y system, and (b) the local x'-y' system.

from the end point i towards the end point j. Note that this coordinate system can be obtained from the xy-system by a rotation of angle α.

From Figures 2 and 3 (b) and Equation (15) we obtain

$$
\begin{aligned}
F'_{x_i} &= -N = k(u'_i - u'_j) \\
F'_{x_j} &= N = k(u'_j - u'_i),
\end{aligned}
\tag{16}
$$

which can be written as

$$
\mathbf{F}^{e'} =
\begin{pmatrix}
F'_{x_i} \\
F'_{y_i} \\
F'_{x_j} \\
F'_{y_j}
\end{pmatrix}
= \mathbf{K}^{e'}\mathbf{u}^{e'} = k
\begin{pmatrix}
1 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 \\
-1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
u'_i \\
v'_i \\
u'_j \\
v'_j
\end{pmatrix},
\tag{17}
$$

where $\mathbf{K}^{e'}$ is the so-called local element stiffness matrix, that is symmetric. It can be observed that the displacements along the y'-axis, i.e., v'_i and v'_j, produce no forces on the bar. This assumption is valid only if the displacements are small; otherwise, the components v'_i and v'_j might result in an elongation of the bar and thus in the development of forces.

Let us now establish a relation between the displacements u_i, v_i, u_j, v_j and u'_i, v'_i, u'_j, v'_j. By geometrical arguments it follows directly that

$$
\begin{aligned}
u_i &= u'_i \cos\alpha - v'_i \sin\alpha \\
v_i &= u'_i \sin\alpha + v'_i \cos\alpha
\end{aligned}
\tag{18}
$$

and the corresponding equations for u_j and v_j.

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Therefore

$$
\mathbf{u}^e = \begin{pmatrix} u_i \\ v_i \\ u_j \\ v_j \end{pmatrix} = \mathbf{L}^{e^T} \mathbf{u}^{e'} = \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 & 0 \\ \sin\alpha & \cos\alpha & 0 & 0 \\ 0 & 0 & \cos\alpha & -\sin\alpha \\ 0 & 0 & \sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} u_i' \\ v_i' \\ u_j' \\ v_j' \end{pmatrix}, \qquad (19)
$$

where $\mathbf{L}^e$ is the so-called transformation matrix, that is an orthogonal matrix $(\mathbf{L}^{e^{-1}} = \mathbf{L}^{e^T})$.

Premultiplying (19) by $\mathbf{L}^e$ and using the orthogonal property, we then conclude that

$$
\mathbf{u}^{e'} = \mathbf{L}^e \mathbf{u}^e. \qquad (20)
$$

As both forces and displacements are vector quantities, we can choose the same basis for the forces and then, the element forces $\mathbf{F}^e$ and $\mathbf{F}^{e'}$ are related in exactly the same manner as the element displacements, i.e.,

$$
\mathbf{F}^e = \mathbf{L}^{e^T} \mathbf{F}'^e. \qquad (21)
$$

With these preliminary remarks, we can determine the relation between the element forces $\mathbf{F}^e$ and the element displacements $\mathbf{u}^e$. Insertion of (20) into (17) yields

$$
\mathbf{F}^{e'} = \mathbf{K}^{e'} \mathbf{L}^e \mathbf{u}^e, \qquad (22)
$$

and premultiplication by $\mathbf{L}^{e^T}$ and using (21) leads to

$$
\mathbf{F}^e = \mathbf{L}^{e^T} \mathbf{F}'^e = \mathbf{L}^{e^T} \mathbf{K}^{e'} \mathbf{L}^e \mathbf{u}^e = \mathbf{K}^e \mathbf{u}^e, \qquad (23)
$$

where $\mathbf{K}^e$ is the so-called global element stiffness matrix. Its explicit expression is

$$
\mathbf{K}^e = k \begin{pmatrix} \cos^2\alpha & \cos\alpha\sin\alpha & -\cos^2\alpha & -\cos\alpha\sin\alpha \\ \cos\alpha\sin\alpha & \sin^2\alpha & -\cos\alpha\sin\alpha & -\sin^2\alpha \\ -\cos^2\alpha & -\cos\alpha\sin\alpha & \cos^2\alpha & \cos\alpha\sin\alpha \\ -\cos\alpha\sin\alpha & -\sin^2\alpha & \cos\alpha\sin\alpha & \sin^2\alpha \end{pmatrix}, \qquad (24)
$$

which shows the symmetric character of the global element stiffness matrix $\mathbf{K}^e$.

The truss analysis problem has the following elements to be considered:

Bars: The longitudinal elements supporting only tension or compression stresses (the number of bars is b).

Nodes: The frictionless elements that join the bars allowing relative rotation between them (the number of hinges or nodes is m).

Supports: Elements that prevent the structure to experiment solid rigid displacements. The reactions exerted by the supports are the necessary ones for the equilibrium to hold. The number of constraints associated with the supports is denoted c.

Forces: The forces acting on the nodes (hinges) of the structure that are the data of our problem.

Displacements: Movements suffered by the nodes owing to the strains produced by the stresses in the bars.

Unknowns: The hinge (nodal) displacements. The number of unknowns coincides with the number of possible hinge displacements, which is the number of degrees of freedom (two times the number of nodes m in case of bi-dimensional analysis).

Equations: The mathematical relations that give the nodal forces in terms of the nodal displacements. To derive the system of equations that model a given problem the global structure stiffness matrix must be obtained.
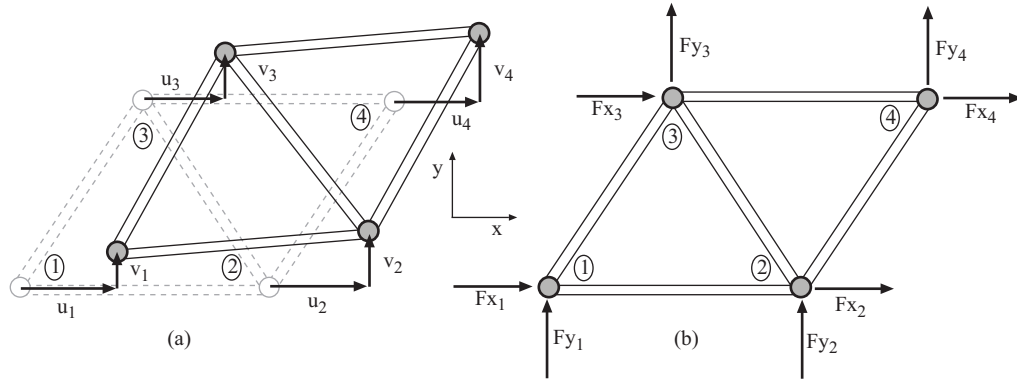

Fig. 4. Displacements and force components for the two-dimensional truss structure.

Example 4 (A simple truss structure) As an example of how the element stiffness matrix can be used to derive the stiffness matrix for the whole structure, consider the simple truss structure shown in Figure 1. Using the relationships between forces and movements, establishing the equilibrium of vertical and horizontal forces for each node, and replacing the static magnitudes (stresses) by their equivalents as a function of the nodal displacements, then the so-called global assembled stiffness matrix is obtained. Note that each element stiffness matrix ($\mathbf{K}^e$) is assembled into the global element stiff-

ness matrix ($\mathbf{K}$), to get the following system of equations:

$$\mathbf{Ku}=\frac{k}{150}\left(\begin{array}{rr|rr|rr|rr} 179 & 72 & -125 & 0 & -54 & -72 & 0 & 0 \\ 72 & 96 & 0 & 0 & -72 & -96 & 0 & 0 \\ \hline -125 & 0 & 233 & 0 & -54 & 72 & -54 & -72 \\ 0 & 0 & 0 & 192 & 72 & -96 & -72 & -96 \\ \hline -54 & -72 & -54 & 72 & 233 & 0 & -125 & 0 \\ -72 & -96 & 72 & -96 & 0 & 192 & 0 & 0 \\ \hline 0 & 0 & -54 & -72 & -125 & 0 & 179 & 72 \\ 0 & 0 & -72 & -96 & 0 & 0 & 72 & 96 \end{array}\right)\left(\begin{array}{c} u_1 \\ v_1 \\ -- \\ u_2 \\ v_2 \\ -- \\ u_3 \\ v_3 \\ -- \\ u_4 \\ v_4 \end{array}\right) = \left(\begin{array}{c} F_{x_1} \\ F_{y_1} \\ -- \\ F_{x_2} \\ F_{y_2} \\ -- \\ F_{x_3} \\ F_{y_3} \\ -- \\ F_{x_4} \\ F_{y_4} \end{array}\right) = \mathbf{F}$$

(25)

where $u_i, v_i; i = 1, \ldots, 4$ are the node displacements shown in Figure 4 (a) and $F_{x_i}, F_{y_i}; i = 1, \ldots, 4$ are the external forces acting on the hinges of the structure including support reactions (see Figure 4 (b)). The different partitions refer to different nodes.

First, using the method described in Section 2.1 the conditions to be satisfied by the linear system (25) to have solution (one or more) are obtained:

$$0 = F_{x_1} + F_{x_2} + F_{x_3} + F_{x_4} \tag{26}$$
$$0 = F_{y_1} + F_{y_2} + F_{y_3} + F_{y_4} \tag{27}$$
$$0 = 4/5 F_{x_1} - 3/5 F_{y_1} + 4/5 F_{x_2} + 3/5 F_{y_2} + 6/5 F_{y_4}, \tag{28}$$

where (26) and (27) express the equilibrium of horizontal and vertical forces respectively, and (28) establishes the equilibrium of moments (the moment are taken with respect to node 3). Thus, these conditions state that the structure subject to the external forces and the support reactions must be in equilibrium ($\sum F_{x_i} = 0, \sum F_{y_i} = 0$ and $\sum M_i = 0$).

The rank of $\mathbf{K}$ in (25) is 5 that implies that three $(8 - 5)$ rows are linear combinations of the other rows, and then the system (25) has infinite solutions. So, some extra equations of the form $\mathbf{Bu} = \mathbf{b}$, where $\mathbf{b}$ is the boundary conditions vector, can be added maintaining the compatibility of system (25).

To obtain the set of all possible solutions of system (25), we apply the orthog-

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onalization algorithm and obtain:

$$
\begin{pmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{pmatrix} = \frac{1}{k} \begin{pmatrix} 467/120 \\ 501/160 \\ 359/120 \\ -359/160 \\ 39/10 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \rho_1 \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + \rho_2 \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} + \rho_3 \begin{pmatrix} 4/5 \\ -3/5 \\ 4/5 \\ 3/5 \\ 0 \\ 0 \\ 0 \\ 6/5 \end{pmatrix} ; \quad \rho_1, \rho_2, \rho_3 \in \mathbb{R},
$$

(29)

which from a mathematical point of view is an affine space, i.e., the sum of a given vector (the first one) plus a linear space of dimension 3 (an arbitrary linear combination of 3 linearly independent vectors).
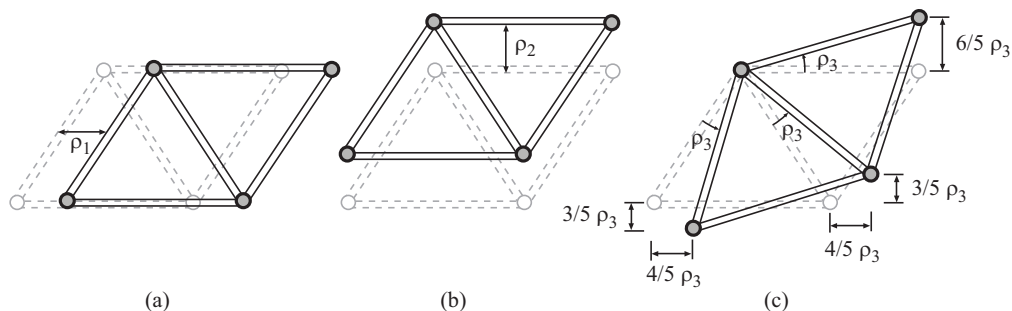

Fig. 5. Illustration of the basis of the linear space of dimension three that appears in the general solution of the two-dimensional truss structure in Example 5. The basis has three generators that correspond to: (a) a horizontal translation, (b) a vertical translation, and (c) a rotation.

Note that the linear space is generated by three vectors that, due to the symmetry of the stiffness matrix, coincide with those in (26)-(28). From an engineering point of view, this solution must be interpreted as follows:

(1) The dimension of the linear space is the number of degrees of freedom we have, i.e., the maximum number of u and v displacements that can be fixed for the system to have a unique solution.
(2) The linear space component of the solution (unlimited values of the ρ coefficients, and consequently unlimited displacements) implies that the structure can be located anywhere whereas the forces acting on the structure guarantee equilibrium.
(3) The first vector is a particular solution, i.e., a solution to the stated problem. Note that it satisfies Equation (25) for the $\mathbf{F}$ values in Figure 6. It is worth noting that this vector can be replaced by any other particular

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solution, which can be obtained by adding to it any linear combination of the three basic vectors.

(4) The second vector corresponds to a solution of the associated homogeneous problem, i.e., with no external forces. In this particular case it corresponds to the horizontal displacement of the structure as a rigid solid (see Figure 5(a)).

(5) The third vector corresponds to another solution of the associated homogeneous problem. In this case, it is the vertical displacement of the structure as a rigid solid (see Figure 5(b)).

(6) Finally, the fourth vector corresponds to another solution of the associated homogeneous problem, which corresponds to the rotation of the structure as a rigid solid with respect node 3 (see Figure 5(c)).

It is obvious that the linear space generated by the last three vectors in (29) can be represented using another basis of the same space, i.e., considering equilibrium with respect other point.

4   Classification of truss structures

Structures can be classified in different ways depending on the rank of the matrix $\left(\mathbf{K} \mid \mathbf{B} \right)^T$ and the value of $b + c$ and m, as shown in Table 1. They include isostatic, hyperstatic, critical and mechanism truss structures:

Isostatic: In this kind of structures $2m = b + c$ and the rank $\left(\mathbf{K} \mid \mathbf{B} \right)^T = 2m$, i.e., a unique solution exists. They are characterized because (a) the boundary conditions ensure equilibrium under all possible external forces, (b) thermic strains do not induce stresses, and (c) if any of their elements (nodes, bars, or boundary conditions) is removed the structure becomes a mechanism or a critical structure.

Hyperstatic: In this kind of structures $b + c > 2m$ where $(b + c) - 2m$ is the degree of hyperstaticity and the rank $\left(\mathbf{K} \mid \mathbf{B} \right)^T = 2m$, i.e., a unique solution exists. They are characterized because (a) the boundary conditions ensure equilibrium under all possible external forces, (b) thermic strains induce stresses, and (c) if any of their elements (nodes, bars, or boundary conditions) is removed the structure becomes a isostatic structure or remains hyperstatic. Note that there are two hyperstatic cases, one is due to an excess of bars in the truss structure, and the other is due to an excess in the supports (boundary conditions).

Mechanism: In this kind of structures $b + c \leq 2m$ and the rank $\left(\mathbf{K} \mid \mathbf{B} \right)^T < 2m$, i.e., infinite solution exists. They are characterized because (a) the

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Table 1
A classification of truss structures.

	Rank $\begin{pmatrix} \mathbf{K} \\ - \\ \mathbf{B} \end{pmatrix}$	$b + c$
Isostatic	$2m$	$2m$
Hyperstatic	$2m$	$> 2m$
Critical	$< 2m$	$\geq 2m$
Mechanism	$< 2m$	$\leq 2m$

boundary conditions do not ensure equilibrium under all possible external forces, (b) thermic strains do not induce stresses.

Critical: In this kind of structures $2m \geq b+c$ and the rank $\left(\mathbf{K} \mid \mathbf{B} \right)^T < 2m$, i.e., infinite solution exists. They are characterized because (a) the boundary conditions do not ensure equilibrium under all possible external forces within the hypothesis of small deformations, (b) thermic strains induce stresses, and (c) if any of their elements (nodes, bars, or boundary conditions) is removed the structure becomes a mechanism.

To illustrate their main differences we include some examples.

Example 5 (An isostatic truss structure) Consider the particular case shown in Figure 6, where there are 3 external forces. Since the vector of external forces $\mathbf{F}$ has to fulfill the compatibility conditions (26)-(28), we use them to obtain the truss reactions:
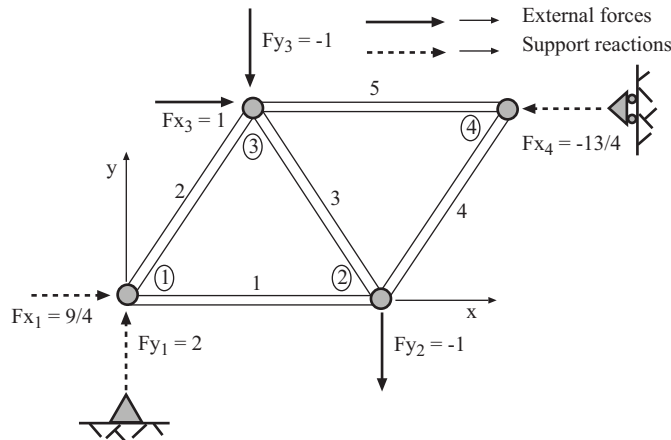

Fig. 6. The two-dimensional truss structure showing the external forces and reactions.

$$F_{x_1} = 9/4; \quad F_{y_1} = 2; \quad F_{x_4} = -13/4;$$

and then, the force vector becomes: $\mathbf{F} = 1/4 \left(9\ 8\ 0\ -4\ 4\ -4\ -13\ 0 \right)^T$.

Next, we consider the boundary conditions imposed by the supports that establish the final location of the structure. Assuming that those boundary conditions are $u_1 = 0, v_1 = 0$ and $u_4 = 0$, then from (29) we have:

$$u_1 = 467/(120k) + \rho_1 + 4/5\rho_3 = 0$$
$$v_1 = 501/(160k) + \rho_2 - 3/5\rho_3 = 0$$
$$v_4 = \rho_2 + 6/5\rho_3 = 0.$$

The solution of this system of equations is $\rho_1 = 0$, $\rho_2 = -121/(20k)$ and $\rho_3 = -467/(96k)$. Thus the corresponding solution obtained after replacing these values of ρ_1, ρ_2 and ρ_3 into (29), is unique and equal to:

$$
\begin{pmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{pmatrix} = \frac{1}{k} \begin{pmatrix} 0 \\ 0 \\ -9/10 \\ -897/80 \\ 39/10 \\ -121/120 \\ 0 \\ -951/80 \end{pmatrix}.
$$

Note that the uniqueness of solution implies that the imposed boundary conditions are enough to avoid rigid solid movements and therefore the supports would be able to react to external forces acting on the structure in such a way that the compatibility conditions (equilibrium) hold. The solution is unique because for the submatrix corresponding to ρ coefficients and u_1, v_1 and v_4 in (29), we have:

$$
\text{Rank} \begin{pmatrix} 1 & 0 & 4/5 \\ 0 & 1 & -3/5 \\ 0 & 1 & 6/5 \end{pmatrix} = 3 \quad \Leftrightarrow \quad \text{rank} \begin{pmatrix} \mathbf{K} \\ - \\ \mathbf{B} \end{pmatrix} = 2m = 8.
$$

Example 6 (An isostatic truss structure) Consider the structure shown in Figure 7 subject to three forces P_1, P_2 and P_3. Applying the superposition principle and assuming that the forces acting on the structure are:

$$\mathbf{F}_1 = P_1 \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 \end{pmatrix}^T,$$

$$\mathbf{F}_2 = P_2 \begin{pmatrix} 3/4 & 1 & 0 & 0 & 0 & -1 & -3/4 & 0 \end{pmatrix}^T$$

$$\mathbf{F}_3 = P_3 \begin{pmatrix} 3/2 & 1 & 0 & -1 & 0 & 0 & -3/2 & 0 \end{pmatrix}^T$$

$$\mathbf{F}_4 = P_4 \begin{pmatrix} 9/4 & 1 & 0 & 0 & 0 & 0 & -9/4 & -1 \end{pmatrix}^T$$

that satisfy the compatibility conditions (26)-(28), the associated system of equations leads to the general solution:

$$
\begin{pmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{pmatrix}
= \frac{P_1}{k} \begin{pmatrix} \frac{3}{5} \\ \frac{9}{20} \\ \frac{3}{5} \\ -\frac{9}{20} \\ \frac{6}{5} \\ 0 \\ 0 \\ 0 \end{pmatrix}
+ \frac{P_2}{k} \begin{pmatrix} \frac{9}{20} \\ \frac{19}{10} \\ \frac{9}{20} \\ -\frac{27}{80} \\ \frac{9}{10} \\ 0 \\ 0 \\ 0 \end{pmatrix}
+ \frac{P_3}{k} \begin{pmatrix} \frac{341}{210} \\ \frac{25}{32} \\ \frac{233}{120} \\ -\frac{233}{160} \\ \frac{9}{5} \\ 0 \\ 0 \\ 0 \end{pmatrix}
+ \frac{P_4}{k} \begin{pmatrix} \frac{287}{60} \\ -\frac{27}{40} \\ \frac{179}{60} \\ -\frac{27}{40} \\ \frac{9}{5} \\ 0 \\ 0 \\ 0 \end{pmatrix}
+ \rho_1 \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}
+ \rho_2 \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}
+ \rho_3 \begin{pmatrix} \frac{4}{5} \\ -\frac{3}{5} \\ \frac{4}{5} \\ \frac{3}{5} \\ 0 \\ 0 \\ 0 \\ \frac{6}{5} \end{pmatrix},
$$

(30)

where $P_1, P_2, P_3, P_4, \rho_1, \rho_1, \rho_3 \in \mathbb{R}$. From a mathematical point of view this is a linear space, i.e., the solutions are the linear combinations of seven given vectors.
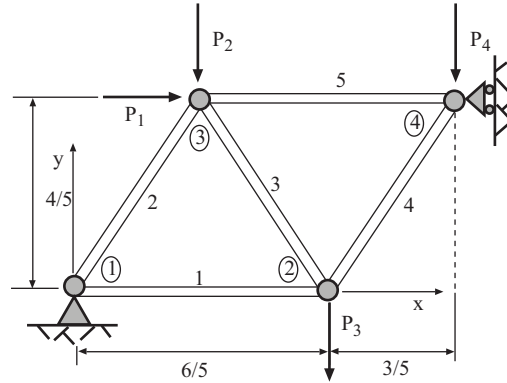

Fig. 7. Isostatic structure in Example 6 showing the applied loads.

Example 7 (A critical truss structure) If we consider the same structure as in Example 5 with other boundary conditions, for example, $u_1 = 0, v_1 = 0$ and $u_2 = 0$, then from (29) we have:

$$u_1 = 467/(120k) + \rho_1 + 4/5\rho_3 = 0$$
$$v_1 = 501/(160k) + \rho_2 - 3/5\rho_3 = 0$$
$$u_2 = 359/(120k) + \rho_1 + 4/5\rho_3 = 0,$$

which is an incompatible system of equations, because for the submatrix associated with the ρ's and u_1, v_1 and u_4 in (29) we have

$$\text{Rank} \begin{pmatrix} 1 & 0 & 4/5 \\ 0 & 1 & -3/5 \\ 1 & 0 & 4/5 \end{pmatrix} = 2 < 3.$$

Adding the equations imposed by the boundary conditions to the system (25), considering the external forces (see Figure 8) $\mathbf{F} = 1/8 \begin{pmatrix} 9 & 12 & 0 & -8 & 8 & -8 & -17 & 4 \end{pmatrix}^T$ that satisfy the compatibility conditions (26)-(28)(equilibrium conditions) and applying the orthogonalization algorithm the following set of all possible solutions is obtained:

$$\begin{pmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{pmatrix} = \frac{1}{k} \begin{pmatrix} 0 \\ 0 \\ 0 \\ -49/16 \\ -1/24 \\ -37/16 \\ -73/24 \\ 0 \end{pmatrix} + \rho \begin{pmatrix} 0 \\ 0 \\ 0 \\ 6/5 \\ -4/5 \\ 3/5 \\ -4/5 \\ 9/5 \end{pmatrix}. \tag{31}$$

From an engineering point of view, this solution must be interpreted as follows:

(1) The first vector is a particular solution, i.e., a solution to the stated problem. Note that it satisfies equation (25) for the $\mathbf{F}$ values considered and the boundary conditions $u_1 = 0, v_1 = 0$ and $u_2 = 0$.

(2) The second vector corresponds to a solution of the associated homogeneous problem, and represents the rotation of the structure as a rigid solid with respect node 1, as it is shown in Figure 8. This implies that there is a set of infinite solutions (rotations), and that the actual boundary conditions are not able to avoid the rigid solid displacement of the structure without the development of new forces in the bars under the hypothesis of small displacements. Moreover, the supports are not capable
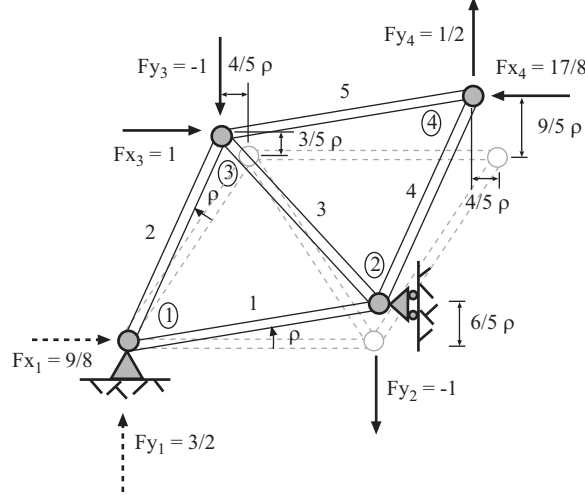
Fig. 8. Illustration of the set of general solutions associated with the boundary conditions $u_1 = 0, v_1 = 0$ and $u_2 = 0$ for the critical two-dimensional truss example.

of supplying the reactions required to satisfy the equilibrium conditions for all possible forces acting on the nodes of the structure.
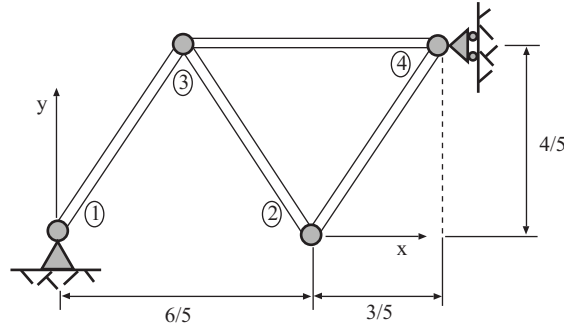
Fig. 9. The mechanism structure analyzed in Example 8.

The same result is obtained considering

$$\mathbf{B} = \begin{pmatrix} 1\ 0\ 0\ 0\ 0\ 0\ 0\ 0 \\ 0\ 1\ 0\ 0\ 0\ 0\ 0\ 0 \\ 0\ 0\ 1\ 0\ 0\ 0\ 0\ 0 \end{pmatrix} \quad \Leftrightarrow \quad \text{rank}\begin{pmatrix} \mathbf{K} \\ - \\ \mathbf{B} \end{pmatrix} = 2m - 1 = 7$$

that implies that there is one rigid solid rotation (ρ) allowed.

Example 8 (A Mechanism structure) Consider the simple mechanism truss structure shown in Figure 9. The system of equations resulting from assem-

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bling the global element stiffness matrices becomes:

$$
\mathbf{Ku} = \frac{k}{150}
\begin{pmatrix}
54 & 72 & 0 & 0 & -54 & -72 & 0 & 0 \\
72 & 96 & 0 & 0 & -72 & -96 & 0 & 0 \\
0 & 0 & 108 & 0 & -54 & 72 & -54 & -72 \\
0 & 0 & 0 & 192 & 72 & -96 & -72 & -96 \\
-54 & -72 & -54 & 72 & 233 & 0 & -125 & 0 \\
-72 & -96 & 72 & -96 & 0 & 192 & 0 & 0 \\
0 & 0 & -54 & -72 & -125 & 0 & 179 & 72 \\
0 & 0 & -72 & -96 & 0 & 0 & 72 & 96
\end{pmatrix}
\begin{pmatrix}
u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8
\end{pmatrix}
=
\begin{pmatrix}
F_{x_1} \\ F_{y_1} \\ F_{x_2} \\ F_{y_2} \\ F_{x_3} \\ F_{y_3} \\ F_{x_4} \\ F_{y_4}
\end{pmatrix}
\tag{32}
$$

and the corresponding compatibility conditions, using the method described in Section 2.1, become:

$$
\begin{align}
0 &= F_{x_1} + F_{x_2} + F_{x_3} + F_{x_4} \tag{33} \\
0 &= F_{y_1} + F_{y_2} + F_{y_3} + F_{y_4} \tag{34} \\
0 &= 4/5 F_{x_1} - 3/5 F_{y_1} \tag{35} \\
0 &= 4/5 F_{x_2} + 3/5 F_{y_2} + 6/5 F_{y_4}, \tag{36}
\end{align}
$$

where (33) and (34) express the equilibrium of horizontal and vertical forces, respectively, and (35) and (36) establish the equilibrium of moments with respect to node 3 of the left and right substructures, respectively.

Assuming that the forces acting on the structure satisfy the compatibility conditions (33)-(36) and take values: $\mathbf{F} = 1/8 \begin{pmatrix} 9 & 12 & 0 & -8 & 8 & -8 & -17 & 4 \end{pmatrix}^T$, the system (32) leads to the general solution:

$$
\begin{pmatrix}
u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4
\end{pmatrix}
= \frac{1}{k}
\begin{pmatrix}
49/8 \\ 0 \\ 3/2 \\ -61/32 \\ 3 \\ 0 \\ 0 \\ 0
\end{pmatrix}
+ \rho_1
\begin{pmatrix}
1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0
\end{pmatrix}
+ \rho_2
\begin{pmatrix}
0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1
\end{pmatrix}
+ \rho_3
\begin{pmatrix}
4/5 \\ -3/5 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0
\end{pmatrix}
+ \rho_4
\begin{pmatrix}
0 \\ 0 \\ 4/5 \\ 3/5 \\ 0 \\ 0 \\ 0 \\ 6/5
\end{pmatrix}
, \tag{37}
$$

where $\rho_1, \rho_1, \rho_3, \rho_4 \in \mathbb{R}$. From a mathematical point of view this is an affine space, i.e., the sum of a given vector (the first one) plus a linear space of dimension 4 (an arbitrary linear combination of 4 linearly independent vectors). The graphical interpretation of these four vectors is shown in Figures 10 (a-d),

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i.e., a horizontal translation, a vertical translation and two independent rotations with respect to node 3. Note that eliminating the bar 1-2 allows the bar 1-3 and the right substructure to rotate with respect to node 3 independently. That is why there are two rotation parameters ρ_3 and ρ_4. Note that in the solution (29) corresponding to the structure with this bar, the rotation vector is the sum of the last two vectors in (37).
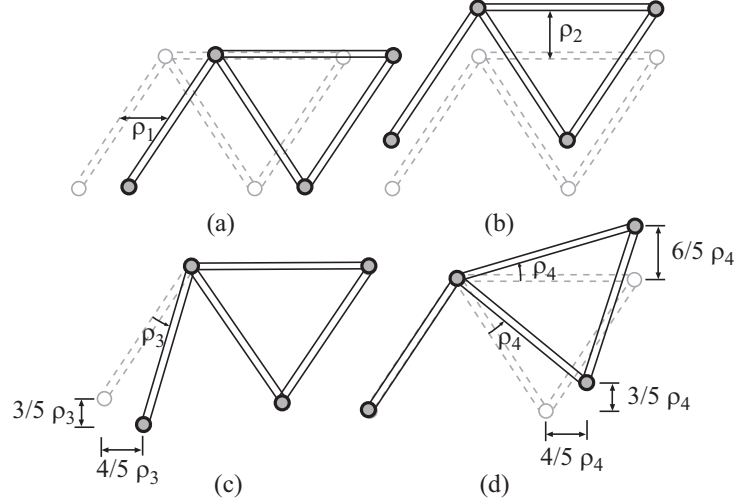
Fig. 10. Illustration of the four basic vectors of the linear space component of the solution corresponding to Example 8. They correspond to a horizontal translation, a vertical translation and two independent rotations with respect to node 3.

If we add the boundary conditions $u_1 = 0$, $v_1 = 0$ and $u_4 = 0$, then from (37) we obtain the system of linear equations:

$$u_1 = 49/(8k) + \rho_1 + 4/5\rho_3 = 0$$
$$v_1 = \rho_2 - 3/5\rho_3 = 0$$
$$u_4 = \rho_1 = 0$$

which solution is $\rho_1 = 0$, $\rho_2 = -147/(32k)$ and $\rho_3 = -245/(32k)$, but ρ_4 becomes free.

Considering the boundary conditions matrix

$$\mathbf{B} = \begin{pmatrix} 1\,0\,0\,0\,0\,0\,0\,0 \\ 0\,1\,0\,0\,0\,0\,0\,0 \\ 0\,0\,0\,0\,0\,0\,1\,0 \end{pmatrix} \quad \Leftrightarrow \quad \mathrm{rank}\begin{pmatrix} \mathbf{K} \\ - \\ \mathbf{B} \end{pmatrix} = 2m - 1 = 7$$

that implies that there is 1 rigid solid movements allowed under the assumption of small displacements without strains.

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Then, the solution in this case becomes:

$$
\begin{pmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{pmatrix} = \frac{1}{k} \begin{pmatrix} 0 \\ 0 \\ -3/2 \\ -13/2 \\ 3 \\ -147/32 \\ 0 \\ -147/32 \end{pmatrix} + \rho_4 \begin{pmatrix} 0 \\ 0 \\ 4/5 \\ 3/5 \\ 0 \\ 0 \\ 0 \\ 6/5 \end{pmatrix},
$$

that implies that the actual boundary conditions do not prevent the rotation of the right substructure with respect node 3 as shown in Figure 11.
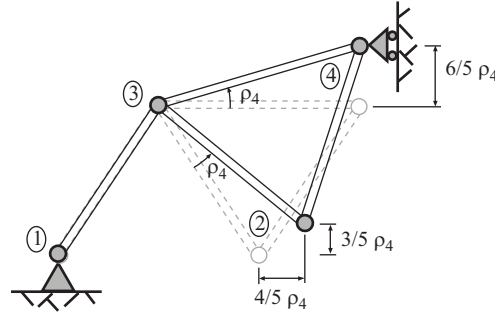
Fig. 11. Illustration of how the boundary conditions $u_1 = 0$, $v_1 = 0$ and $u_4 = 0$ do not prevent the rotation of the right substructure with respect node 3 in a mechanism structure.
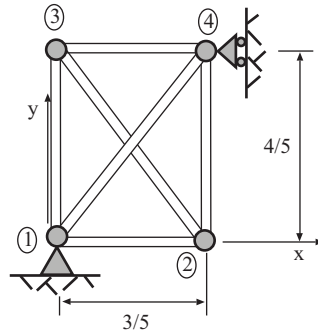
Fig. 12. The hyperstatic structure analyzed in Example 9.

Example 9 (An hyperstatic structure) Consider now the structure given in Figure 12 where an additional bar joining nodes 1 and 4 has been added. The system of equations resulting from assembling the global element stiffness

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matrices in this case is:

$$
\mathbf{Ku} = \frac{k}{300}
\begin{pmatrix}
608 & 144 & -500 & 0 & 0 & 0 & -108 & -144 \\
144 & 567 & 0 & 0 & 0 & -375 & -144 & -192 \\
-500 & 0 & 608 & -144 & -108 & 144 & 0 & 0 \\
0 & 0 & -144 & 567 & 144 & -192 & 0 & -375 \\
0 & 0 & -108 & 144 & 608 & -144 & -500 & 0 \\
0 & -375 & 144 & -192 & -144 & 567 & 0 & 0 \\
-108 & -144 & 0 & 0 & -500 & 0 & 608 & 144 \\
-144 & -192 & 0 & -375 & 0 & 0 & 144 & 567
\end{pmatrix}
\begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \end{pmatrix}
=
\begin{pmatrix} F_{x_1} \\ F_{y_1} \\ F_{x_2} \\ F_{y_2} \\ F_{x_3} \\ F_{y_3} \\ F_{x_4} \\ F_{y_4} \end{pmatrix}
\tag{38}
$$

and the compatibility conditions become:

$$
\begin{align}
0 &= F_{x_1} + F_{x_2} + F_{x_3} + F_{x_4} \tag{39} \\
0 &= F_{y_1} + F_{y_2} + F_{y_3} + F_{y_4} \tag{40} \\
0 &= 4/5 F_{x_1} + 4/5 F_{x_2} + 3/5 F_{y_2} + 3/5 F_{y_4}, \tag{41}
\end{align}
$$

which correspond to the equilibrium of horizontal forces, vertical forces and moments with respect to node 3.

Assuming that the forces acting on the structure satisfy the compatibility conditions (39)-(41) and take values $\mathbf{F} = 1/4 \left(3\ 8\ 0\ -4\ 4\ -4\ -7\ 0\right)^T$, the following general solution is obtained:

$$
\begin{pmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{pmatrix}
= \frac{1}{k}
\begin{pmatrix} 353/960 \\ 521/540 \\ 11/40 \\ -343/540 \\ 133/192 \\ 0 \\ 0 \\ 0 \end{pmatrix}
+ \rho_1
\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}
+ \rho_2
\begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}
+ \rho_3
\begin{pmatrix} 4/5 \\ 0 \\ 4/5 \\ 3/5 \\ 0 \\ 0 \\ 0 \\ 3/5 \end{pmatrix}
; \quad \rho_1, \rho_1, \rho_3 \in \mathbb{R}.
\tag{42}
$$

From a mathematical point of view it is an affine linear space, i.e., the sum of a given vector (the first one) plus a linear space of dimension 3 (an arbitrary linear combination of 3 linearly independent vectors).

Example 10 (Hyperstatic structure) Consider the particular truss structure in Figure 13, subject to 3 external forces. Note that it is the same example as the initial one but adding a new boundary condition ($v_4 = 0$), thus we add a new equation to the system (25) but we have a new unknown variable, the
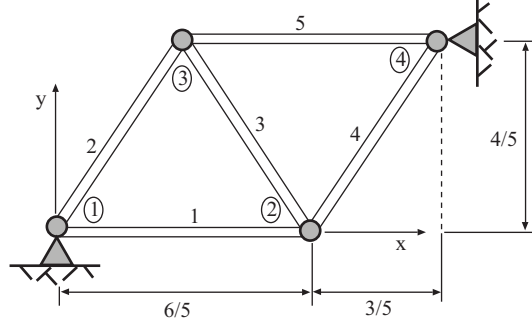
Fig. 13. Hyperstatic truss structure of Example 10.

vertical support reaction on node 4 (F_{y_4}). Applying the superposition principle we transform our problem as it is shown in Figure 14.

The set off all possible solutions will be composed by (29) and the particular solution associated with the vector

$$\mathbf{F} = \frac{1}{4}\begin{pmatrix} -9 & -4 & 0 & 0 & 0 & 0 & 9 & 4 \end{pmatrix}^T,\tag{43}$$

which is

$$\begin{pmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{pmatrix} = \frac{1}{k}\begin{pmatrix} 467/120 \\ 501/160 \\ 359/120 \\ -359/160 \\ 39/10 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \frac{F_{y_4}}{k}\begin{pmatrix} -287/60 \\ 27/40 \\ -179/60 \\ 27/40 \\ -9/5 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \rho_1\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + \rho_2\begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} + \rho_3\begin{pmatrix} 4/5 \\ -3/5 \\ 4/5 \\ 3/5 \\ 0 \\ 0 \\ 0 \\ 6/5 \end{pmatrix},$$

(44)

where $F_{y_4}, \rho_1, \rho_2, \rho_3 \in \mathbb{R}$, that from a mathematical point of view is an affine linear space, i.e., the sum of a given vector (the first one) plus a linear space of dimension 4 (an arbitrary linear combination of 4 linearly independent vectors). The graphical interpretation of the last three vectors is shown in Figure 5, whereas the first one is the particular solution associated with the system of forces (43).

If we add the boundary conditions $u_1 = 0, v_1 = 0, u_4 = 0$ and $v_4 = 0$, then

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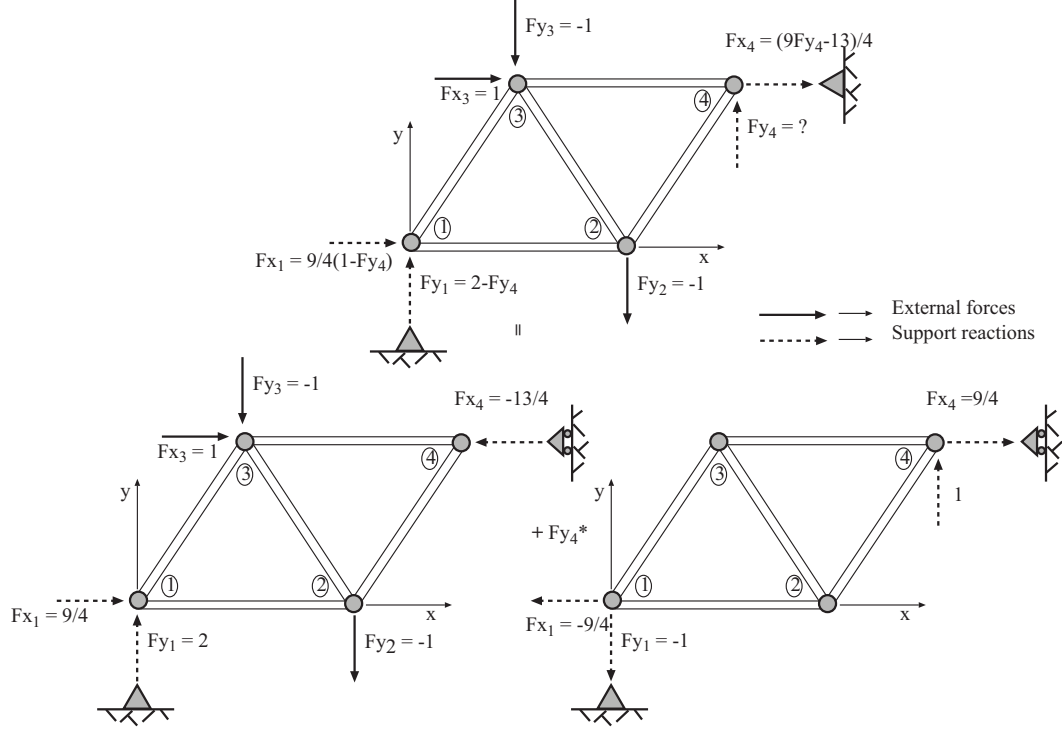
Fig. 14. Illustration of the superposition principle applied to Example 10.

from (38) we have:

$$
\begin{pmatrix} u_1 - 467/(120k) \\ v_1 - 501/(160k) \\ u_4 \\ v_4 \end{pmatrix} = \begin{pmatrix} -287/(60k) & 1 & 0 & 4/5 \\ 27/(40k) & 0 & 1 & -3/5 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 6/5 \end{pmatrix} \begin{pmatrix} F_{y_4} \\ \rho_1 \\ \rho_2 \\ \rho_3 \end{pmatrix}, \qquad (45)
$$

which solution is $F_{y_4} = 317/269$, $\rho_1 = 0$, $\rho_2 = -11267/(4304k)$ and $\rho_3 = 56335/(25824k)$. Thus the solution is unique and equal to:

$$
\begin{pmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{pmatrix} = \frac{1}{k} \begin{pmatrix} 0 \\ 0 \\ 657/538 \\ -11867/4304 \\ 957/538 \\ -11267/4304 \\ 0 \\ 0 \end{pmatrix}.
$$

5   Solving Systems of Inequalities

In many engineering applications we find systems of linear inequalities of the form:

$$\mathbf{A}\mathbf{x} \leq \mathbf{b}. \tag{46}$$

Before trying to solve a system of inequalities of this type it is interesting to check whether or not the system is compatible, that is, if it has some solutions. Even, in many cases we can be interested in obtaining the conditions for the system to be compatible in terms of the independent terms $b_1, b_2, \ldots, b_m$. These conditions always have a physical or engineering meaning that raises some light about the problem under consideration. Once the system is proven to have solution we solve the system and obtain all its possible solutions.

To solve these two problems the Γ algorithm is used (see Jubete [7,8], Padberg [9], Castillo, Jubete, Pruneda and Solares [5], Castillo, Esquivel y Pruneda [6] and Castillo, Conejo, Pedregal, García and Alguacil [4]), that gives the dual cone of a given cone and is the key tool to discuss the compatibility problem and solve the system of inequalities.

Since the concepts of cone and dual cone are used, we start with their definitions.

Definition 1 (Polyhedral convex cone) *Let $\mathbf{A}$ be a matrix, and $\{\mathbf{a}_1, \ldots, \mathbf{a}_m\}$ be its column vectors. The set*

$$\mathbf{A}_\pi \equiv \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{x} = \pi_1\mathbf{a}_1 + \cdots + \pi_m\mathbf{a}_m \quad with \quad \pi_j \geq 0; j = 1, \ldots, m\}$$

of all nonnegative linear combinations of the column vectors of $\mathbf{A}$ is known as the polyhedral convex cone generated by $\mathbf{a}_1, \ldots, \mathbf{a}_m$ (its generators), and is denoted $\mathbf{A}_\pi$.

A cone $\mathbf{A}_\pi$ can be written as the sum of a linear space $\mathbf{V}_\rho$ plus a pure (acute) cone $\mathbf{W}_\pi$, i.e., $\mathbf{A}_\pi = \mathbf{V}_\rho + \mathbf{W}_\pi$.

In this paper we use the Greek letter π to refer to non-negative real numbers.

Definition 2 (Nonpositive dual or polar cone) *Let $\mathbf{A}_\pi$ be a cone in $\mathbb{R}^n$ with generators $\mathbf{a}_1, \ldots, \mathbf{a}_k$. The nonpositive dual of $\mathbf{A}_\pi$, denoted $\mathbf{A}_\pi^p$, is defined as the set*

$$\mathbf{A}_\pi^p \equiv \left\{\mathbf{u} \in \mathbb{R}^n \mid \mathbf{A}^T\mathbf{u} \leq \mathbf{0}\right\} \equiv \left\{\mathbf{u} \in \mathbb{R}^n \mid \mathbf{a}_i^T\mathbf{u} \leq 0; \ i = 1, \ldots, k\right\}$$

that is, the set of all vectors such that their dot products by all vectors in $\mathbf{A}_\pi$ are nonpositive.

Definition 3 (Polytope) *Let $\mathbf{A}$ be a matrix, and $\{\mathbf{a}_1, \ldots, \mathbf{a}_m\}$ be its column vectors. The set*

$$\mathbf{A}_\lambda \equiv \left\{ \mathbf{x} \in \mathbb{R}^n \mid \mathbf{x} = \lambda_1 \mathbf{a}_1 + \cdots + \lambda_m \mathbf{a}_m \quad with \quad \lambda_j \geq 0; j = 1, \ldots, m; \sum_{i=1}^{m} \lambda_i = 1 \right\}$$

of all linear convex combinations of the column vectors of $\mathbf{A}$ is known as the polytope generated by $\mathbf{a}_1, \ldots, \mathbf{a}_m$ (its generators), and is denoted $\mathbf{A}_\lambda$.

5.1 Deciding whether or not a system of linear inequalities is compatible

In this section we show how to analyze the compatibility of a system of linear inequalities.

First, we discuss the compatibility of a particular system of the form:

$$
\begin{aligned}
a_{11}x_1 \;+a_{12}x_2 \;&+\; \cdots \;+a_{1n}x_n \;=\; b_1, \\
a_{21}x_1 \;+a_{22}x_2 \;&+\; \cdots \;+a_{2n}x_n \;=\; b_2, \\
\cdots \quad\quad \cdots \quad &\cdots \;\; \cdots \quad\quad\quad \cdots \\
a_{m1}x_1 \;+a_{m2}x_2 \;&+\; \cdots \;+a_{mn}x_n \;=\; b_m \\
&x_1, x_2, \cdots, x_n \;\geq\; 0
\end{aligned}
\tag{47}
$$

that can be written as

$$
x_1 \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix} + x_2 \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix} + \cdots + x_n \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}
\tag{48}
$$
$$x_1, x_2, \cdots, x_n \geq 0$$

Expression (48) shows that the given system is compatible if and only if the vector $\mathbf{b} = (b_1, \ldots, b_m)^T$ belongs to the cone generated by the set of column vectors $\{\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_n\}$ of the coefficient matrix $\mathbf{A}$, i.e.,

$$\mathbf{b} \in \mathbf{A}_\pi = \mathbf{b} \in (\mathbf{A}_\pi^p)^p \tag{49}$$

Thus, the compatibility problem reduces to finding the dual cone $\mathbf{V}_\rho + \mathbf{W}_\pi$ of the cone generated by the columns of the coefficient matrix and checking that $\mathbf{b}^T \mathbf{V} = \mathbf{0}$ and $\mathbf{b}^T \mathbf{W} \leq \mathbf{0}$.

To analyze the compatibility of an arbitrary system of linear inequalities, it can be converted to the case in (47), using slack variables to convert the inequalities in equalities, and one more artificial variable to convert the arbitrary variables into no negative variables, that is, each variable x_i can be converted to $x_i^* - x_0$, where $x_0, x_i^* \geq 0$.

Example 11 (Compatibility of a linear system of equations in restricted variables)
Consider the following linear system:

$$
\begin{aligned}
x_2 - x_3 - 2x_4 &= b_1 \\
x_3 + x_4 &= b_2 \\
-x_2 + x_3 + 2x_4 &= b_3 \\
x_2 + x_3 - x_4 &= b_4 \\
-x_1 + 2x_2 + x_3 + x_4 &= b_5 \\
x_1, x_2, x_3, x_4 &\geq 0.
\end{aligned}
\tag{50}
$$

For it to be compatible, the vector $\mathbf{b} = (b_1, b_2, b_3, b_4, b_5)^T$ must belong to the cone $\mathbf{C}_\pi$ generated by the columns of the coefficient matrix, that is to say, it must belong to the dual of the dual of $\mathbf{C}_\pi$. Thus, the compatibility problem reduces to finding the dual cone $\mathbf{C}_\pi^p \equiv \mathbf{V}_\rho + \mathbf{W}_\pi$ and checking that $\mathbf{b}^T \mathbf{V} = \mathbf{0}$ and $\mathbf{b}^T \mathbf{W} \leq \mathbf{0}$.

Since $\mathbf{C}_\pi^p$ is the cone:

$$
\mathbf{C}_\pi^p = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}_\rho + \begin{pmatrix} -2 & 1 & -1 & -4 \\ -3 & 1 & -2 & -7 \\ 0 & 0 & 0 & 0 \\ 1 & -1 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix}_\pi ,
$$

we obtain the desired compatibility conditions:

$$
\begin{pmatrix} b_1 & b_2 & b_3 & b_4 & b_5 \end{pmatrix} \begin{pmatrix} -2 & 1 & -1 & -4 \\ -3 & 1 & -2 & -7 \\ 0 & 0 & 0 & 0 \\ 1 & -1 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix} \leq \mathbf{0},
\tag{51}
$$

$$
\begin{pmatrix} b_1 & b_2 & b_3 & b_4 & b_5 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 & 0 & 0 \end{pmatrix}^T = b_1 + b_3 = 0.
$$

Consider the homogeneous system of inequalities

$$\mathbf{A}\mathbf{x} \leq \mathbf{0} \tag{52}$$

which can be written as

$$\mathbf{a}^i \mathbf{x}^T \leq 0; \ i = 1, \ldots, m. \tag{53}$$

Expression (53) shows that $(x_1, \ldots, x_n)$ is the dual cone of the row vectors $\{\mathbf{a}^1, \mathbf{a}^1, \ldots, \mathbf{a}^m\}$ of $\mathbf{A}$.

Thus, obtaining the solution of the system (52) reduces to determining the dual cone $\mathbf{A}_\pi^p$ of the cone generated by the cone generated by the rows of matrix $\mathbf{A}$.

Thus, the general solution of an homogeneous system of linear inequalities is a cone, that is, its general solution is of the form

$$\mathbf{x} = \sum_{i=1}^p \rho_i \mathbf{v}_i + \sum_{j=1}^q \pi_j \mathbf{w}_j; \quad \rho_i \in \mathbb{R}; \quad \pi_j \in \mathbb{R}^+.$$

Example 12 (Solving an homogeneous system of linear inequalities)
Consider the system of equations

$$
\begin{aligned}
-x_5 &\leq 0 \\
x_1 + x_2 \quad\quad + x_4 - x_5 &\leq 0 \\
-x_1 + x_2 - 2x_3 - x_4 + x_5 &\leq 0 \\
-2x_1 \quad\quad + x_3 - x_4 - x_5 &\leq 0 \\
2x_1 + x_2 \ - x_3 \quad\quad + x_5 &\leq 0.
\end{aligned}
\tag{54}
$$

To solve this system, we obtain the dual cone of the cone generated by the rows coefficients and obtain the solution:

$$
\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix}
=
\begin{pmatrix}
2 & 2 & 0 & -2 & -2 \\
-5 & -1 & -1 & -1 & 3 \\
-1 & 3 & -1 & -1 & 3 \\
-5 & -1 & 1 & 3 & 3 \\
0 & 0 & 0 & 0 & 4
\end{pmatrix}
\begin{pmatrix} \pi_1 \\ \pi_2 \\ \pi_3 \\ \pi_4 \\ \pi_5 \end{pmatrix}
; \quad \pi_i \in \mathbb{R}^+; \ i = 1, 2, \ldots, 5.
$$

5.3   Solving a complete system of linear inequalities

Now consider the complete system of linear inequalities:

$$\mathbf{A}\mathbf{x} \leq \mathbf{b} \tag{55}$$

where $\mathbf{A} \in \mathbb{R}^{m \times n}$ and $\mathbf{b} \in \mathbb{R}^m$.

Adding the artificial variable x_{n+1}, the constraint $x_{n+1} = 1$ and the redundant constraint $x_{n+1} \geq 0$ (it is a key trick that allows the constraint $x_{n+1} = 1$ to be easily forced at the end of the process), it can be written as

$$
\begin{aligned}
a_{11}x_1 \ +a_{12}x_2 \ +\cdots \ +a_{1n}x_n \ -b_1x_{n+1} \ &\leq \ 0 \\
a_{21}x_1 \ +a_{22}x_2 \ +\cdots \ +a_{2n}x_n \ -b_2x_{n+1} \ &\leq \ 0 \\
\cdots \quad\cdots \quad\cdots \quad\cdots \quad\quad\cdots \quad\cdots& \\
a_{m1}x_1 \ +a_{m2}x_2 \ +\cdots \ +a_{mn}x_n \ -b_mx_{n+1} \ &\leq \ 0 \\
a_{m1}x_1 \ +a_{m2}x_2 \ +\cdots \ +a_{mn}x_n \ -b_mx_{n+1} \ &\leq \ 0 \\
-x_{n+1} \ &\leq \ 0 \\
x_{n+1} \ &= \ 1.
\end{aligned}
\tag{56}
$$

System (56) can be written as

$$
\begin{aligned}
(a_{11},\ldots,a_{1n},-b_1)(x_1,\ldots,x_n,x_{n+1})^T \ &\leq \ 0 \\
(a_{21},\ldots,a_{2n},-b_2)(x_1,\ldots,x_n,x_{n+1})^T \ &\leq \ 0 \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \quad \cdots& \\
(a_{m1},\ldots,a_{mn},-b_m)(x_1,\ldots,x_n,x_{n+1})^T \ &\leq \ 0 \\
-x_{n+1} \ &\leq \ 0 \\
x_{n+1} \ &= \ 1.
\end{aligned}
\tag{57}
$$

Expression (57) shows that $(x_1,\ldots,x_n,x_{n+1})$ belongs to the dual cone of the cone generated by the set of vectors

$$\{(a_{11},\ldots,a_{1n},-b_1),(a_{21},\ldots,a_{2n},-b_2),\ldots,(a_{m1},\ldots,a_{mn},-b_m),(0,0,\cdots,0,-1)\}.$$

Then, it is clear that the solution of (56) is the intersection of that cone with the hyperplane $x_{n+1} = 1$. Thus, the solution of (55) is the projection on $X_1 \times \cdots \times X_n$ of the solution of (56). In other words, the general solution of

a complete system of linear inequalities is a polyhedron, that is the sum of a linear space, a cone and a polytope, that is, its general solution is of the form

$$\mathbf{x} = \sum_{i=1}^{p} \rho_i \mathbf{v}_i + \sum_{j=1}^{q} \pi_j \mathbf{w}_i + \sum_{k=1}^{r} \lambda_k \mathbf{u}_k; \quad \rho_i \in \mathbb{R}, \ \pi_j \in \mathbb{R}^+; \ \lambda_k \in \mathbb{R}^+; \sum_{k=1}^{r} \lambda_k = 1.$$

Example 13 (Solving a complete system of linear inequalities) To solve the following system of inequalities:

$$
\begin{array}{rrrrrr}
x_1 & +x_2 & & +x_4 & \leq & 1 \\
-x_1 & +x_2 & -2x_3 & -x_4 & \leq & -1 \\
-2x_1 & & +x_3 & -x_4 & \leq & 1 \\
2x_1 & +x_2 & -x_3 & & \leq & -1,
\end{array}
\tag{58}
$$

we use the auxiliary variable x_5 and the redundant constraint $1 = x_5 \geq 0$. Then, the system (58) can be written as:

$$
\begin{array}{rrrrrr}
& & & -x_5 & \leq & 0 \\
x_1 & +x_2 & & +x_4 & -x_5 & \leq & 0 \\
-x_1 & +x_2 & -2x_3 & -x_4 & +x_5 & \leq & 0 \\
-2x_1 & & +x_3 & -x_4 & -x_5 & \leq & 0 \\
2x_1 & +x_2 & -x_3 & & +x_5 & \leq & 0 \\
& & & & x_5 & = & 1.
\end{array}
\tag{59}
$$

Since the upper part is an homogeneous system, one need to find the dual cone of the cone generated by the row coefficients. After imposing condition $x_5 = 1$ one gets the solution:

$$
\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -1/2 \\ 3/4 \\ 3/4 \\ 3/4 \end{pmatrix} + \begin{pmatrix} 2 & 2 & 0 & -2 \\ -5 & -1 & -1 & -1 \\ -1 & 3 & -1 & -1 \\ -5 & -1 & 1 & 3 \end{pmatrix} \begin{pmatrix} \pi_1 \\ \pi_2 \\ \pi_3 \\ \pi_4 \end{pmatrix},
\tag{60}
$$

which has no linear space part and whose polytope part reduce to a single vector.

6   The Truss Analysis Problem Revisited

In this section we analyze the truss analysis problem, but assuming additional constraints related to node displacements and maximum bar compressions.
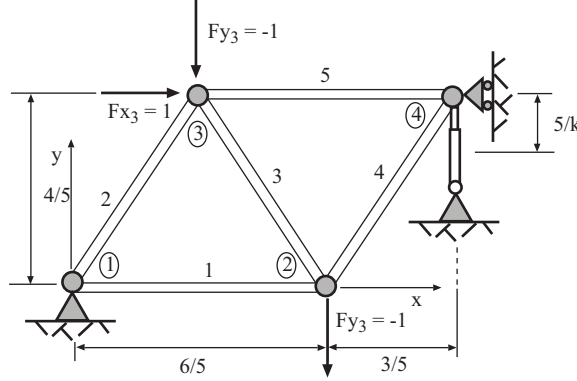

Fig. 15. Truss structure of Example 14 showing the acting forces and the support conditions.

Example 14 (The lifting jack structure) Consider the structure shown in Figure 15 where there is a lifting jack that push the node 4 in the vertical direction. We want to obtain the minimum force that has to be applied by the lifting jack, for the descent of node 4 to be smaller than $5/k$. We would like to get the set of all possible node displacements for all the valid values of the force exerted by the lifting jack.

As the boundary conditions in this case are $u_1 = 0$, $v_1 = 0$, $u_4 = 0$ and $v_4 \geq -5/k$, from (44) we can get the following system of linear inequalities:

$$u_1 - 467/(120k) = -287/(60k)F_{y_4} + \rho_1 + 4/5\rho_3$$
$$v_1 - 501/(160k) = 27/(40k)F_{y_4} + \rho_2 - 3/5\rho_3$$
$$u_4 = \rho_1$$
$$5/k \geq -\rho_2 - 6/5\rho_3$$

(61)

which solution, using the Γ-algorithm is:

$$\begin{pmatrix} F_{y_4} \\ \rho_1 \\ \rho_2 \\ \rho_3 \end{pmatrix} = \begin{pmatrix} 551/807 \\ 0 \\ -52441/(12912k) \\ -60595/(77472k) \end{pmatrix} + \pi \begin{pmatrix} 1 \\ 0 \\ 233/(80k) \\ 287/(48k) \end{pmatrix} ; \quad \pi \geq 0, \qquad (62)$$

which is composed of a particular solution and a cone generated by a single vector. Note that $F_{y_4} = 551/807$ is the force necessary to get a vertical

displacement of node 4 equal to $-5/k$. Since it cannot be smaller and must satisfy the constraint $v_4 \geq -5/k$, the value of π has to be positive.

Replacing (62) in (44) we get:

$$
\begin{pmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{pmatrix} = \frac{1}{k} \begin{pmatrix} 0 \\ 0 \\ 177/538 \\ -81521/12912 \\ 1437/538 \\ -52441/12912 \\ 0 \\ -5 \end{pmatrix} + \frac{\pi}{k} \begin{pmatrix} 0 \\ 0 \\ 9/5 \\ 287/40 \\ -9/5 \\ 233/80 \\ 0 \\ 807/80 \end{pmatrix} ; \quad \pi \geq 0, \tag{63}
$$

which, as before, is the sum of a particular solution and a cone generated by a single vector.

From an engineering point of view, this solution must be interpreted as follows:

(1) The particular solution gives the displacements produced by the force $F_{y_4} = 551/807$ that is the minimum force required to satisfy the constraint $v_4 \geq -5/k$. Note that the vertical displacement of node 4 is equal to $-5/k$.

(2) The cone generator represents the displacement increments produced by a unit force acting on the lifting jack. Note that the vertical displacement at node 4 is equal to $807/(80k)$, and that because the π-value has to be positive the vertical displacement of node 4 is always greater than $-5/k$.

(3) Note that both solutions satisfy the boundary conditions $u_1 = 0$, $v_1 = 0$, $u_4 = 0$.

Example 15 (Maximum compression constraint) Consider the same structure as in Figure 7. The aim is to find all possible values of forces P_1, P_2, and P_3, in such a way that bars 1 and 5 are always subject to positive stresses (tractions). To solve this problem a relation between the displacements $u_i^{(l)}$, $v_i^{(l)}$, $u_j^{(l)}$, $v_j^{(l)}$ and the axial force $N^{(l)}$ acting on bar l must be established. From

(15) and (20) we have:

$$\mathbf{N}^{(l)} = k \left(-1\ 0\ 1\ 0 \right) \mathbf{L}^{(l)} \mathbf{u}^{(l)} = k \left(-\cos\alpha^{(l)}\ -\sin\alpha^{(l)}\ \cos\alpha^{(l)}\ \sin\alpha^{(l)} \right) \begin{pmatrix} u_i^{(l)} \\ v_i^{(l)} \\ u_j^{(l)} \\ v_j^{(l)} \end{pmatrix}. \tag{64}$$

If bars 1 and 5 are to be subject to tensions ($N^{(1)} \geq 0$ and $N^{(5)} \geq 0$) and the structure must satisfy the boundary conditions $u_1 = 0$, $v_1 = 0$ and $u_4 = 0$, the following system of inequations obtained from (30) and (64) must be solved:

$$u_1 = 3P_1/(5k) + 9P_2/(20k) + 341P_3/(120k) + 287P_4/(60k) + \rho_1 + 4/5\rho_3$$

$$v_1 = 9P_1/(20k) + 19P_2/(10k) + 25P_3/(32k) - 27P_4/(40k) + \rho_2 - 3/5\rho_3$$

$$u_4 = \rho_1$$

$$0 \geq 3P_3/4 + 3P_4/2$$

$$0 \geq P_1 + 3P_2/4 + 3P_3/2 + 3P_4/2, \tag{65}$$

where the last two inequalities represent the constraints $-N^{(1)} \leq 0$ and $-N^{(5)} \leq 0$, respectively.

The solution of the system using the Γ-algorithm is:

$$\begin{pmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \\ \rho_1 \\ \rho_2 \\ \rho_3 \end{pmatrix} = \rho_4 \begin{pmatrix} 3/4 \\ 1 \\ -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \rho_5 \begin{pmatrix} \dfrac{12}{25} \\ -\dfrac{16}{25} \\ 0 \\ 0 \\ 0 \\ \dfrac{1}{k} \\ 0 \end{pmatrix} + \pi_1 \begin{pmatrix} 1 \\ 4/3 \\ -4/3 \\ 0 \\ 0 \\ 0 \\ \dfrac{233}{(72k)} \end{pmatrix} + \pi_2 \begin{pmatrix} -\dfrac{179}{125} \\ \dfrac{72}{125} \\ 0 \\ 0 \\ 0 \\ 0 \\ \dfrac{3}{(4k)} \end{pmatrix} \; ; \rho_4, \rho_5 \in \mathbb{R}; \; \pi_1, \pi_2 \geq 0 \tag{66}$$

Replacing (66) in (30) we get:

$$
\begin{pmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{pmatrix} = \frac{\rho_4}{k} \begin{pmatrix} 0 \\ 0 \\ 0 \\ \dfrac{25}{16} \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \frac{\rho_5}{k} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} + \frac{\pi_1}{k} \begin{pmatrix} 0 \\ 0 \\ 1 \\ \dfrac{179}{60} \\ 0 \\ 0 \\ 0 \\ \dfrac{233}{60} \end{pmatrix} + \frac{\pi_2}{k} \begin{pmatrix} 0 \\ 0 \\ 0 \\ \dfrac{9}{10} \\ -\dfrac{6}{5} \\ 0 \\ 0 \\ \dfrac{9}{10} \end{pmatrix} \; ; \rho_4, \rho_5 \in \mathbb{R}\,; \pi_1, \pi_2 \geq 0,
$$

(67)

which is a cone, i.e., the sum of a linear space of dimension 2 and an acute cone generated by two vectors.

From an engineering point of view, this solution must be interpreted as follows:

(1) The fist linear space generator in (67) corresponds to a vertical displacement of node 2 with the remaining nodes being fixed, which leads to compressions in bars 3 and 4 and no stresses in the remaining bars (see Figure 16(a)).

(2) The second linear space generator in (67) corresponds to a vertical displacement of nodes 2, 3 and 4 with node 1 remaining fixed, which corresponds to a rigid vertical displacement of the substructure defined by those nodes, while bar 2 is subject to tension stress. This implies that bars 1, 3, 4 and 5 are subject to no stress (see Figure 16(b)).

(3) The first acute cone generator in (67) corresponds to a rotation with respect to node 3 of the substructure defined by nodes 2, 3 and 4, with the bar 2 remaining fixed, while bar 1 is subject to tension stress. This implies that bars 2, 3, 4 and 5 are subject to no stress (see Figure 16(c)).

(4) The second acute cone generator in (67) corresponds to a vertical translation of nodes 3 and 4 and a horizontal displacement of node 3. This implies that bar 2 is subject to compression, bar 5 to tension, and bars 1, 3 and 4 are subject to no stress (see Figure 16(d)).

(5) Note that all vertices satisfy the boundary conditions $u_1 = v_1 = u_4 = 0$.

(6) Note also that the rotation, horizontal and vertical displacements associated with the acute cone generators must have the sign indicated in Figures 16(c) and 16(d) ($\pi_1, \pi_2 \in \mathbb{R}^+$) for the bars 1 and 5 being subject to positive stress (tension), while the vertical displacements associated with the linear space generators can have any direction ($\rho_4, \rho_5 \in \mathbb{R}$) because bars 1 and 5 do not work under these displacements.
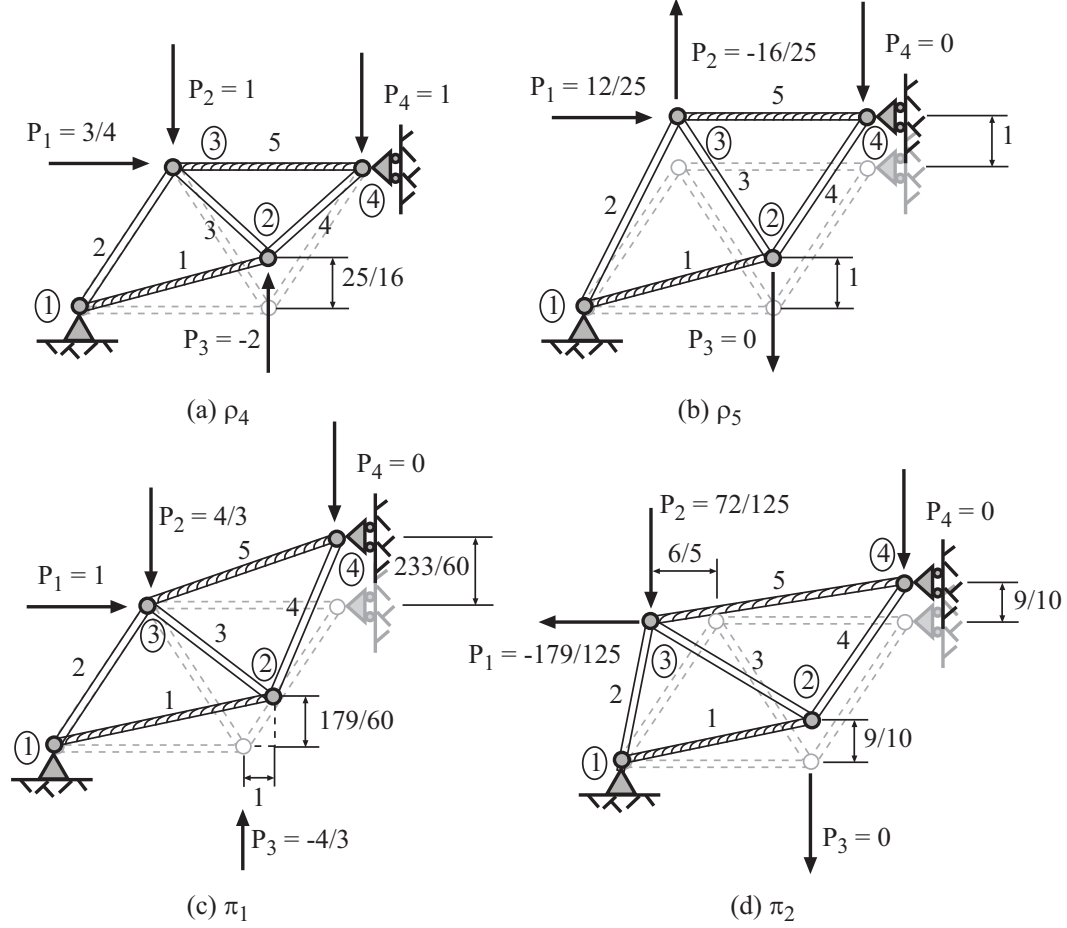
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Fig. 16. Illustration of the different basic vectors that generate the linear space and the cone components of the solution of Example 15, where the tension bars 1 and 5 have been distinguished from the remaining bars.

Example 16 (Tension, compression and deflection constraints) Consider the same structure as in Figure 7, but now bars 3 and 4 have been replaced by cables, that can only stand tensions, and bars $1, 2$ and 5 are built using concrete, material specially suitable to support compressions.

The target of this example consists of obtaining the values of the forces P_1, P_2, P_3 and P_4 that can be applied on the structure in such a way that the cables work under tension and the concrete beams under compression, respectively. In addition, the maximum vertical deflection of node 4 is limited to $v_2 \geq -1/k$.

If bars 3, and 4 are to be subject to tensions ($N^{(3)} \geq 0$ and $N^{(4)} \geq 0$), bars $1, 2$ and 5 to compressions ($N^{(1)}, N^{(2)} \leq 0$ and $N^{(5)} \leq 0$) and the structure must satisfy the boundary conditions $u_1 = 0$, $v_1 = 0$ and $u_4 = 0$, the following

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system of inequations obtained from (30) and (64) must be solved:

$$u_1 = 3P_1/(60k) + 9P_2/(20k) + 341P_3/(120k) + 287P_4/(60k) + \rho_1 + 4/5\rho_3$$

$$v_1 = 9P_1/(20k) + 19P_2/(10k) + 25P_3/(32k) - 27P_4/(40k) + \rho_2 - 3/5\rho_3$$

$$u_4 = \rho_1$$

$$0 \geq -3P_3/4 - 3P_4/4$$

$$0 \geq -5P_2/4 - 5P_3/4 - 5P_4/4$$

$$0 \geq -5P_3/4 - 5P_4/4$$

$$0 \geq 5P_4/4$$

$$0 \geq -P_1 - 3P_2/4 - 3P_3/2 - 3P_4/2$$

$$1/k \geq 9P_1/20 + 27P_2/80 + 233P_3/160 + 27P_4/40 - \rho_2 - 3/5\rho_3,$$

(68)

where the last six inequalities represent the constraints $N^{(1)} \leq 0$, $N^{(2)} \leq 0$, $-N^{(3)} \leq 0$, $-N^{(4)} \leq 0$, $N^{(5)} \leq 0$ and $-v_2 \leq 1/k$, respectively.

The solution of the system using the Γ-algorithm is:

$$
\begin{pmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \\ \rho_1 \\ \rho_2 \\ \rho_3 \end{pmatrix} = \lambda_1 \begin{pmatrix} -\dfrac{12}{25} \\ \dfrac{16}{25} \\ \dfrac{32}{25} \\ -\dfrac{16}{25} \\ 0 \\ 0 \\ 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} -\dfrac{12}{15} \\ \dfrac{16}{25} \\ 0 \\ 0 \\ 0 \\ -1/k \\ 0 \end{pmatrix} + \lambda_3 \begin{pmatrix} \dfrac{5}{9} \\ 0 \\ 0 \\ 0 \\ 0 \\ -\dfrac{1}{(2k)} \\ -\dfrac{5}{(12k)} \end{pmatrix} + \lambda_4 \begin{pmatrix} -\dfrac{60}{179} \\ -\dfrac{80}{179} \\ \dfrac{80}{179} \\ 0 \\ 0 \\ 0 \\ -\dfrac{1165}{(1074k)} \end{pmatrix} + \lambda_5 \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix},
$$

(69)

where

$$\sum_{i=1}^{5} \lambda_i = 1; \ \lambda_i \geq 0; \ i = 1, \ldots, 5,$$

(70)

and replacing (69) in (30) we get:

$$
\begin{pmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{pmatrix} = \frac{\lambda_1}{k} \begin{pmatrix} 0 \\ 0 \\ 0 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \frac{\lambda_2}{k} \begin{pmatrix} 0 \\ 0 \\ 0 \\ -1 \\ 0 \\ -1 \\ 0 \\ -1 \end{pmatrix} + \frac{\lambda_3}{k} \begin{pmatrix} 0 \\ 0 \\ 0 \\ -1 \\ 2/3 \\ -1/2 \\ 0 \\ -1 \end{pmatrix} + \frac{\lambda_4}{k} \begin{pmatrix} 0 \\ 0 \\ -72/179 \\ -1 \\ 0 \\ 0 \\ 0 \\ -233/179 \end{pmatrix} + \frac{\lambda_5}{k} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix},
$$

(71)

where $\sum_{i=1}^{5} \lambda_i = 1$; $\lambda_i \geq 0$; $i = 1, \ldots, 5$, which is the sum of a polytope generated by 5 vectors (vertices).
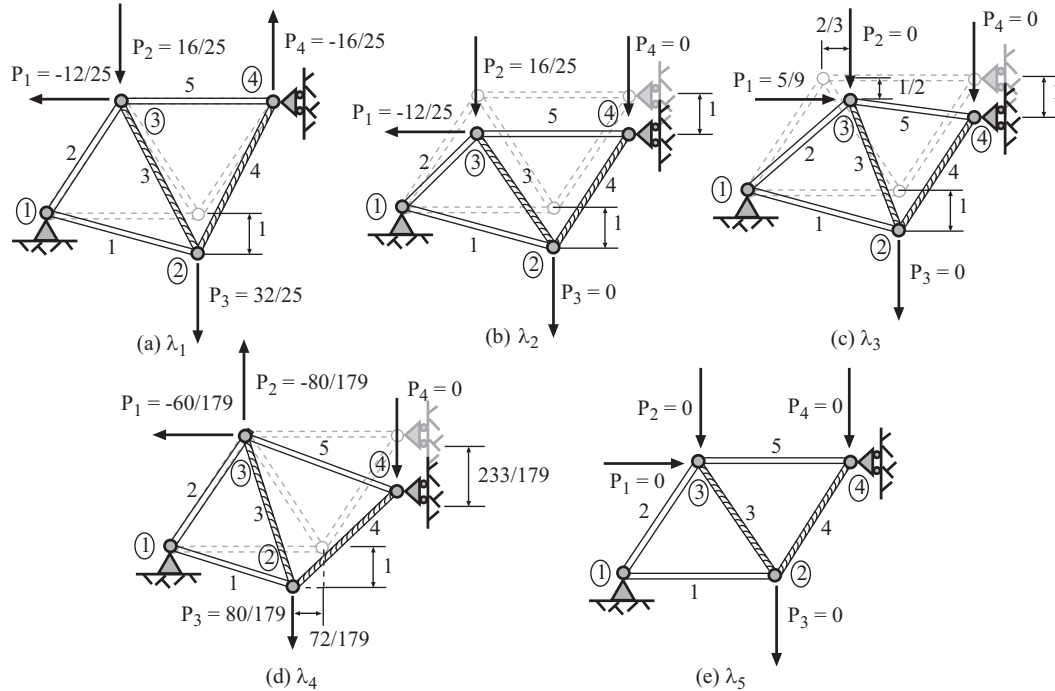

Fig. 17. Illustration of the different basic vectors that generate the linear space and the cone components of the solution of Example 16.

From an engineering point of view, this solution must be interpreted as follows:

(1) The fist vertex in (71) corresponds to a vertical displacement of node 2 with the remaining nodes being fixed, which leads to traction stresses in bars 3 and 4 and no stress in the remaining bars (see Figure 17(a)).
(2) The second vertex in (71) corresponds to a vertical displacement of nodes 2, 3 and 4 with node 1 remaining fixed, which corresponds to a rigid

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vertical displacement of the substructure defined by those nodes, while node 2 is subject to compression stress. This implies that bars 1, 3, 4 and 5 are to no stress (see Figure 17(b)).

(3) The third vertex in (71) corresponds to a vertical displacement of bar 4 and a rotation with respect to node 1 of the substructure defined by nodes 1, 2 and 3, which leads to a compression of bar 5. This implies that bars 1, 2, 3 and 4 are subject to no stress (see Figure 17(c)).

(4) The fourth vertex in (71) corresponds to a rotation with respect to node 3 of the substructure defined by nodes 2, 3 and 4, with the bar 2 remaining fixed, while bar 1 is subject to compression stress. This implies that bars 2, 3, 4 and 5 are subject to no stress (see Figure 17(d)).

(5) The fifth vertex in (71) corresponds to no displacement at all, which implies no stress in all bars (see Figure 17(e)).

(6) Note that all vertices satisfy the boundary conditions $u_1 = v_1 = u_4 = 0$.

(7) Note also that the maximum allowed vertical displacement in node 2 $(v_2 \geq -1/k)$ corresponds to all cases with $\lambda_5 = 0$, i.e., $v_2 = -(\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4) = -1$.

7   Conclusions

The following conclusions can be derived from this paper:

(1) A full understanding of real problems stated as systems of linear equations or inequalities requires both the mathematical and the engineering points of view that complement each other.

(2) The compatibility conditions must be interpreted from an engineering point of view, which help to identify errors, omissions or possible discrepancies between the mathematical model and the reality being modeled.

(3) The mathematical structures of the general solutions, linear spaces, cones, polytopes and mixed combinations of these three structures have clear engineering interpretations that are closely related to the real problem being modeled.

(4) The generators of the solution set, i.e., the linear space generators (basis), the cone generators, and the polytope generators (vertices) have clear interpretations from an engineering point of view, and contains a valuable information on the general solution of the problem and its properties.

References

[1] E. Castillo, A. Cobo, A. Fernandez-Canteli, F. Jubete, and R. E. Pruneda. Updating inverses in matrix analysis of structures, *Internat. J. Numer. Methods*

Engrg. **31**, No. 1, 43, pp. 1479–1504, 1998.

[2] E. Castillo, A. Cobo, F. Jubete, and R. E. Pruneda. *Orthogonal Sets and Polar Methods in Linear Algebra: Applications to Matrix Calculations, Systems of Equations and Inequalities, and Linear Programming,* John Wiley, New York, 1999.

[3] E. Castillo, A. Cobo, F. Jubete, R. E. Pruneda and C. Castillo. An Orthogonally Based Pivoting Transformation of Matrices and Some Applications, *SIAM Journal on Matrix Analysis and Applications* 22, no.3, 666-681, 2000.

[4] E. Castillo, A. Conejo, P. Pedregal, R. García and N. Alguacil. *Building and Solving Mathematical Programming Models in Engineering and Science,* New York: John Wiley & Sons, 2001.

[5] E. Castillo, F. Jubete, E. Pruneda and C. Solares. Obtaining simultaneous solutions of linear subsystems of equations and inequalities, *Linear Algebra and its Applications*, 346, 131-154, 2002.

[6] E. Castillo, M. Esquivel and R. E. Pruneda. Automatic Generation of Linear Programming Problems for Computer Aided Instruction, *International Journal of Mathematical Education in Science and Technology*, 32, 209-232, 2001.

[7] F. Jubete. *El cono poliédrico convexo. Su incidencia en el álgebra lineal y la programación no lineal,* Editorial CIS, Santander, Spain, 1991.

[8] F. Jubete. *El Politopo. Su estructura geométrica y volumen exacto,* Editorial CIS, Santander, Spain, 1993.

[9] M. Padberg. *Linear optimization and extensions,* Springer, Berlín, 1995.